Trapezoid courses

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The typical Trapezoid course

The typical *Trapezoid*[1] course shown in Figure 1 is two windward-leeward courses parallel to each other and designed to accommodate two different classes or two fleets of the same class, on the same course area and using the same start and finish lines.

The reaching leg between marks 1 and 2 is there as a 'spacer' between the *Inner* and *Outer* loops and the length of the reaching leg should be approximately two-thirds of the length of the windward leg (the beat). The angle θ is chosen so that the reach is a 'tight reach', say $\theta = 70^\circ$ for non-spinnaker yachts and $\theta = 60^\circ$ for yachts carrying spinnakers, and the windward legs of both courses are the same length.

The start line (S) is 100-150m to leeward of mark 4, the finish line (F) is at the end of a short reach of 250-300m from mark 3, and all marks are left to port.

Yachts on the *Outer trapezoid* start first and sail the course: **S–1–2–3–2–3–F** (the dotted line in Figure 1).

Yachts on the *Inner trapezoid* start after the first fleet and sail the course: **S–1–4–1–2–3–F** (the full line in Figure 1)

If the two fleets are different classes of yachts with different potential speeds, then the course for the faster fleet can be lengthened by adding additional windward/return legs with the aim of yachts in both fleets having similar elapsed times for their races.

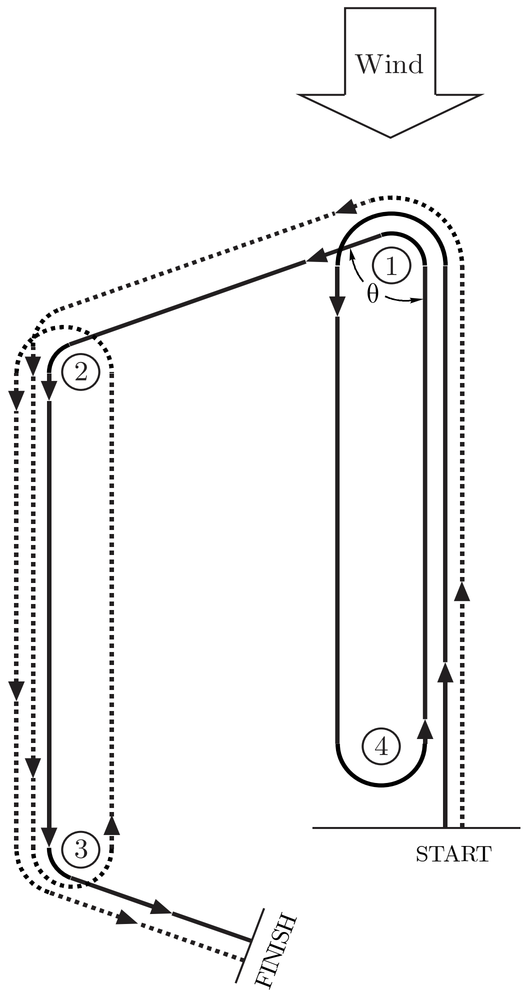

Figure 1. Trapezoid course.

[1] In these notes a *trapezoid* (North American English) or *trapezium* (British English) are both taken to mean a four-sided figure having *at least* one pair of parallel sides. This definition includes a *parallelogram,* a *rectangle,* and a *square* as special cases of a trapezoid.

Trapezoid course with beat to finish

Another type of trapezoid course that I call *Trapezoid course with beat to finish* is shown in Figure 2, with the two windward-leeward courses parallel to each other but with a second reaching leg between marks 3 and 5 followed by a short windward leg to the start/finish line.

The reaching legs 1–2 3–5 (both the same length) act as spacers between the *Inner* and *Outer* loops and the length of these legs should be approximately two-thirds of the length of the windward leg 1–4.

The angle θ at mark 1 is the same as the angle at mark 5 and the angles at marks 2 and 3 are both $180 - \theta$.

The windward legs of both courses can be different lengths or the same lengths with $2 - 3 \leq 1 - 4$.

The start line (S) is the same as the finish line (F) and is 100-150m to leeward of mark 4 and all marks are left to port.

Yachts on the *Outer trapezoid* start first and sail the course: **S–1–2–3–2–3–5–F** (the dotted line in Figure 2).

Yachts on the *Inner trapezoid* start after the first fleet and sail the course: **S–1–4–1–2–3–5–F** (the full line in Figure 2)

Courses can be lengthened or shortened by choosing different length windward legs or lengthened by adding additional windward/return legs.

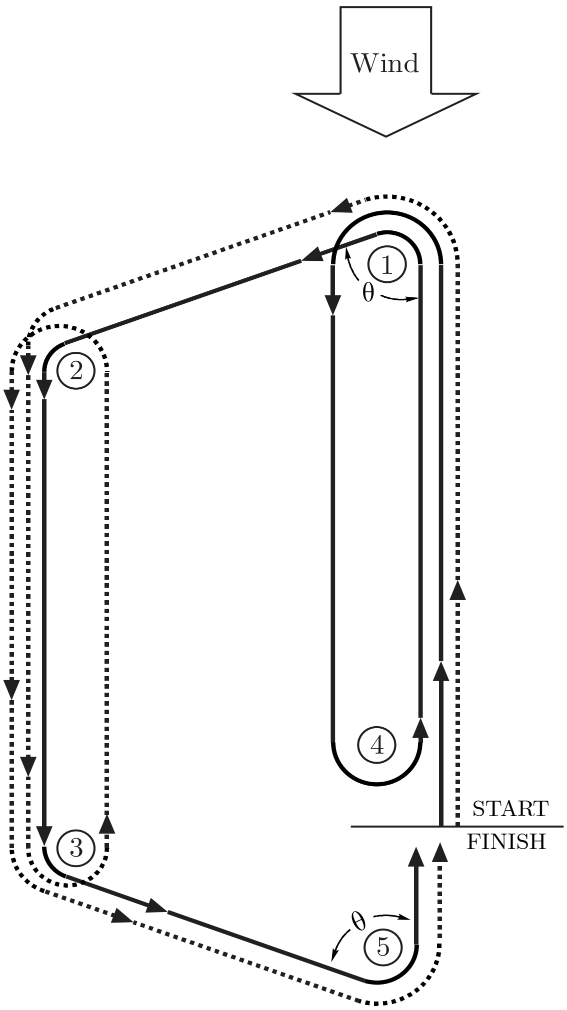

Figure 2: Trapezoid course with beat to finish.

Calculating course lengths and mark laying data

For the purposes of calculating course lengths and mark laying data consider Figure 3 where the wind is from the North and the course bearing from mark 4 to mark 1 is $0° \, 00'$ and since the two windward/leeward courses are parallel the bearing from mark 2 to 3 is $180° \, 00'$. Mark 5 is directly downwind of 4 and the start/finish line is approximately 100 m downwind of mark 4. The figure 1-2-3-5 is a trapezoid.

The distances of the two windward/leeward legs are a and b with $b \leq a$ and the lengths of both reaching legs are ka where $k \leq 1$ is a multiplying factor. World Sailing (2024) use $k = 2/3$ in their Race Management Manual.

The x,y coordinate system with origin at mark 4

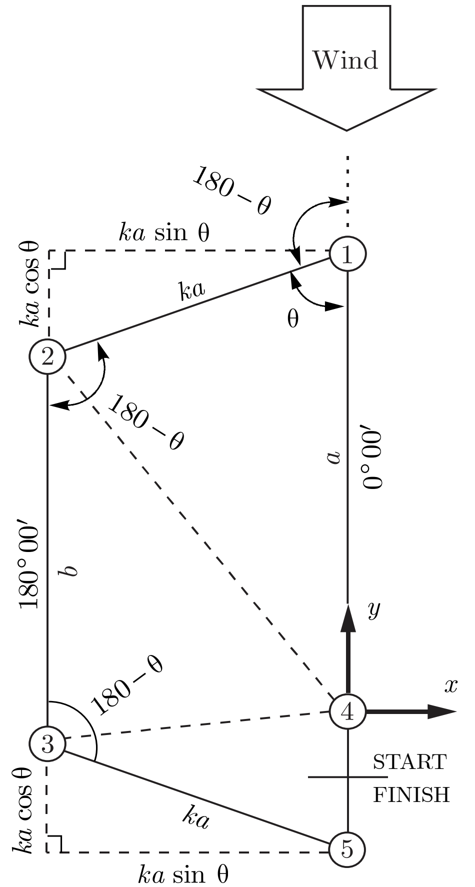

Figure 3. Trapezoid course diagram.

In Figure 3, mark 4 is a reference point and the origin of an x-y rectangular coordinate system. Each leg of the course can be resolved into x and y components that are denoted dx and dy shown in Table 1

Leg	dx	dy
4-1	0	a
1-2	$-ka\sin\theta$	$-ka\cos\theta$
2-3	0	$-b$
3-5	$ka\sin\theta$	$-ka\cos\theta$
5-4	0	$b + 2ka\cos\theta - a$

Table 1.

The x,y coordinates of each mark are shown in Table 2

Mark	x	y
4	0	0
1	0	a
2	$-ka\sin\theta$	$a - ka\cos\theta$
3	$-ka\sin\theta$	$a - ka\cos\theta - b$
5	0	$a - 2ka\cos\theta - b$

Table 2.

The *E,N* coordinate system with origin at mark 4

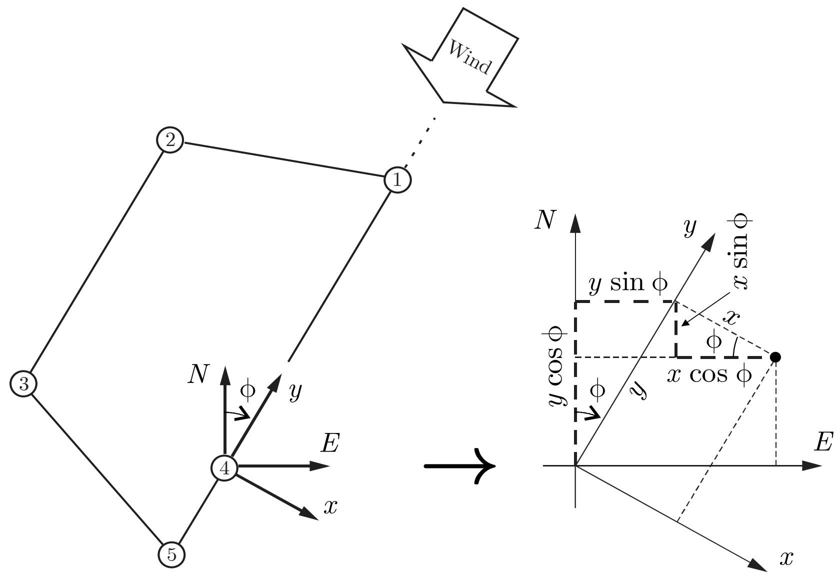

Figure 4. Trapezoid course rotated clockwise from North.

The *E,N* coordinates system (E = east, N = north) and the *x,y* coordinate system have a common origin at mark 4 with the *x,y* system rotated by an angle ϕ . ϕ is the Greek letter phi and denotes the wind direction, measured positive clockwise from the North axis, and in sailing, this is usually Magnetic North.

Figure 4 shows the relationship between *E,N* and *x,y* coordinates that can be expressed mathematically as

$$\begin{aligned} E_j &= x_j \cos\phi + y_j \sin\phi \\ N_j &= y_j \cos\phi - x_j \sin\phi \end{aligned} \tag{1}$$

Where the subscript $j = 1,2,3,4,5$ and the x_j, y_j coordinates are given in Table 2. For example the *E,N* coordinates of mark 3 are $E_3 = x_3 \cos\phi + y_3 \sin\phi$ and $N_3 = y_3 \cos\phi - x_3 \sin\phi$

For setting the course on the water it may be useful to know the bearing and distance from the reference mark 4 (the origin O of the *E,N* system) to marks 2 and 3 (shown as dashed lines on Figure 2).

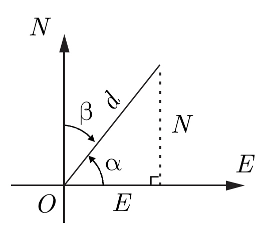

Figure 5

In Figure 5, O is the origin of the *E,N* coordinates (mark 4), β is a bearing measured clockwise from the *N*-axis, d is the distance from the origin and α is an angle measured positive anticlockwise from the *E*-axis.

Distances d_j and angles α_j can be obtained from the following formula where subscripts j = 1,2,3, etc.

Pythagoras' theorem:

$$d_j = \sqrt{E_j^2 + N_j^2} \tag{2}$$

Tangent function

$$\tan\alpha_j = \frac{N_j}{E_j} \tag{3}$$

Inverse tangent function $$\arctan\left(\frac{N_j}{E_j}\right) = \alpha_j \text{ where } -\tfrac{\pi}{2} < \alpha_j < \tfrac{\pi}{2} \tag{4}$$

atan2 function $$\alpha_j = \text{atan2}\left(N_j, E_j\right) \text{ where } -\pi < \alpha_j \le \pi \tag{5}$$

In equations (3), (4) and (5), angle α is given in radians[2] and measured positive anticlockwise and the relationship between the functions tan(), arctan() and atan2() is explained in detail in the Appendix.

To determine the bearings of lines, where a bearing β is defined to be a positive clockwise angle measured from the N-axis (see Figure 5), two useful formula are (see Appendix)

$$\beta = \pi - \alpha - \text{sgn}\left(E+\varepsilon\right)\frac{\pi}{2} = \pi - \arctan\left(\frac{N}{E+\varepsilon}\right) - \text{sgn}\left(E+\varepsilon\right)\frac{\pi}{2} \tag{6}$$

Where sgn is the *signum* or *sign* function and $\text{sgn}\left(x\right) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$, and $\varepsilon = 1E-10$

Or

$$\beta = \pi + \alpha = \pi + \text{atan2}\left(-E, -N\right) \tag{7}$$

The calculations required to compute the coordinates *x,y,* and *E,N*, bearings β and distances *d* can all be done in Microsoft Excel (or other spreadsheet calculation software).

To compute bearings using Excel, equation (6) be written in a cell location as

```
=180-DEGREES(ATAN(C3/(B3+0.0000000001)))-90*SIGN(B3+0.0000000001)
```
(8)

Where `DEGREES()` is an Excel function to convert radians to degrees, `ATAN()` is Excel's inverse tangent function arctan(), `C3` and `B3` are cell locations for *N* and *E* coordinates respectively, the small quantity $\varepsilon = 1E-10 = 0.0000000001$, and `SIGN()` is Excel's signum function.

Alternatively, to compute bearings using Excel, equation (7) can be written in a cell location as

```
=180+DEGREES(ATAN2(-B3,-C3))
```
(9)

Where `ATAN2()` is Excel's atan2() function and `B3` and `C3` are cell locations for *E* and *N* coordinates respectively, noting that Microsoft Excel has the following definition: $\arctan\left(\frac{b}{a}\right) = \text{atan2}\left(a,b\right)$ for $a \ne 0$.

[2] Radians are a circular dimensionless measure of an angle, i.e., 1 radian is the angle subtended at the centre of a circular arc of length equal to the radius, and since the circumference of a unit circle is 2π then $2\pi \text{ radians} = 360^\circ$.

Example course

For the purposes of checking spreadsheet calculations, the following example Trapezoid course may be useful.

In Figure 6, the Trapezoidal course has the following parameters:

$a = 1000$ m	leg length 4-1	$k = 0.7$	ratio (leg 1-2)/(leg 4-1)
$b = 800$ m	leg length 3-2	$\theta = 60°$	course angle at mark 1
		$\phi = 30°$	wind angle

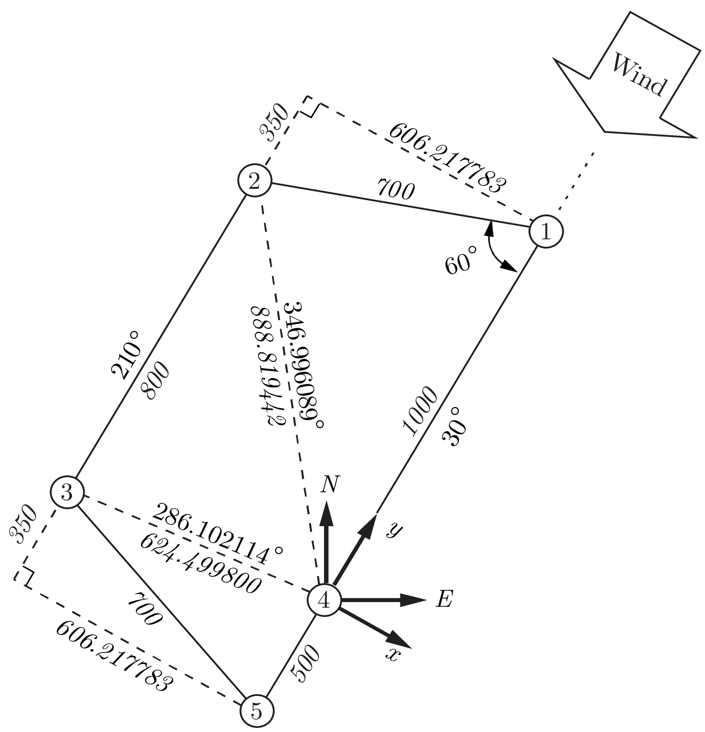

Figure 6. Trapezoidal course for wind angle 30°

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**TRAPEZOID YACHT COURSE**												
2													
3	a =	1000.00	metres		**Leg**	**dx**	**dy**		**Mark**	**x**	**y**	**E**	**N**
4	b =	800.00			4-1	0.000000	1000.000000		4	0.000000	0.000000	0.000000	0.000000
5	theta =	60.00	degress		1-2	-606.217783	-350.000000		1	0.000000	1000.000000	500.000000	866.025404
6	k =	0.70			2-3	0.000000	-800.000000		2	-606.217783	650.000000	-200.000000	866.025404
7	phi =	30.00	degrees		3-5	606.217783	-350.000000		3	-606.217783	-150.000000	-600.000000	173.205081
8					5-4	0.000000	500.000000		5	0.000000	-500.000000	-250.000000	-433.012702
9													
10					**Leg**	**Distance**			**Line**	**Bearing**	**Distance**		
11					4-1	1000.0			4-1	30.000000	1000.000000		
12					1-2	700.0			4-2	346.996088	888.819442		
13					2-3	800.0			4-3	286.102114	624.499800		
14					3-5	700.0			4-5	210.000000	500.000000		
15					5-4	500.0							

Figure 7. Extract from a Microsoft Excel spreadsheet

REFERENCES

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APPENDIX

RECTANGULAR *(x,y)* → POLAR *(r,θ)*

In mathematics, a rectangular *x,y* coordinate system (see Figures A1 and A2 below) divides the plane into four quadrants numbered I, II, III, and IV and a point P in the xy plane has coordinates (x,y) where x is considered as positive along OX and negative along OX' while y is positive along OY and negative along OY'. The lines $X'OX$ and $Y'OY$ are the x and y axis respectively.

The distance from the origin O to P is positive and denoted by $r = \sqrt{x^2 + y^2}$. The angle θ described *anticlockwise* from *OX* is considered *positive.* If it is described *clockwise* from *OX,* it is considered *negative.* In Figure A1, for example, $P(x,y)$ is in the second quadrant and angle θ positive anticlockwise from the x axis while in Figure A2, $P(x,y)$ is in the third quadrant and the angle θ negative clockwise from the x axis.

In mathematics, angles are given in radians[3] and the conversion from radians to degrees (°) is obtained from the relationship π radians $= 180°$

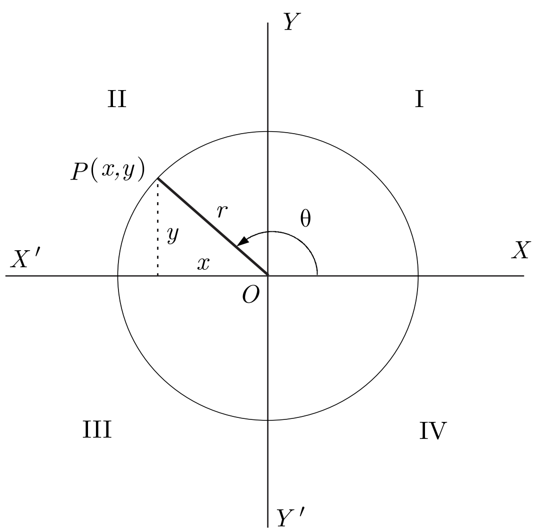

Figure A1

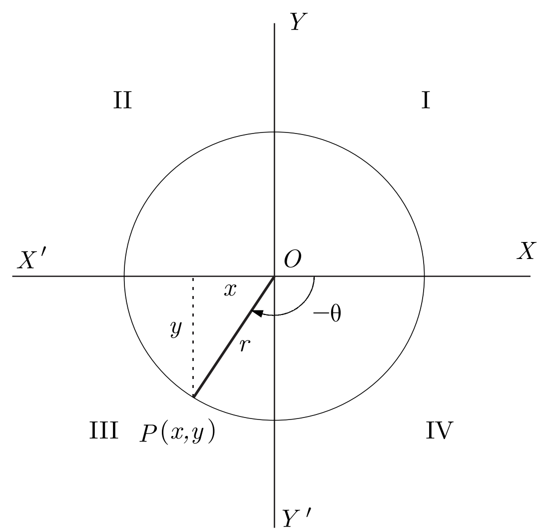

Figure A2

Determining the polar angle *θ*

The trigonometric *tangent* function is defined by

$$\tan\theta = \frac{y}{x} \tag{10}$$

where y and x are the opposite and adjacent sides respectively to angle θ in a right-angle triangle.

[3] Radians are a circular dimensionless measure of an angle, i.e., 1 radian is the angle subtended at the centre of a circular arc of length equal to the radius, and since the circumference of a unit circle is 2π then 2π radians $= 360°$.

The angle θ is given by the *inverse* trigonometric function arctan, and

$$\arctan\left(\frac{y}{x}\right) = \theta \qquad \text{and} \quad -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \tag{11}$$

Note: $\tan\theta$ is undefined when $x = 0$ (division by zero) and to avoid this problem in calculations, a very small value say $\varepsilon = 1E-10$ can be added to x before using the arctan function.

Figure A3 below shows the four cases of $\theta = \arctan\left(\frac{y}{x}\right)$ for $P\left(x,y\right)$ in quadrants I, II, III, and IV.

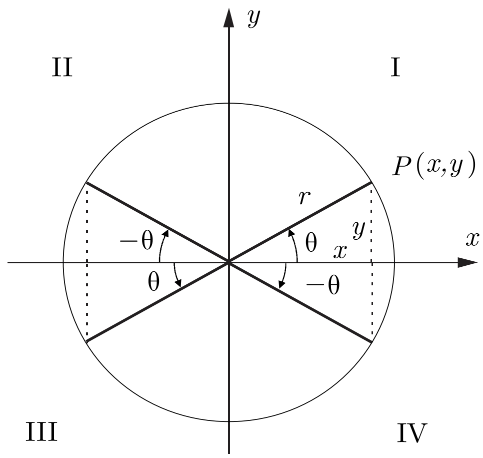

Figure A3. $\theta = \arctan\left(\frac{y}{x}\right)$

Table A1 shows the values of $\theta = \arctan\left(\frac{y}{x}\right)$ for $P\left(x,y\right)$ in quadrants I to IV dependent on positive or negative values of x and y.

Quadrant	$\left(\frac{y}{x}\right)$	$\theta = \arctan\left(\frac{y}{x}\right)$
I	$\left(\frac{+}{+}\right)$	$+\theta$
II	$\left(\frac{+}{-}\right)$	$-\theta$
III	$\left(\frac{-}{-}\right)$	$+\theta$
IV	$\left(\frac{-}{+}\right)$	$-\theta$

Table A1. Valus of $\arctan\left(\frac{y}{x}\right)$ in quadrants

The atan2 function

When converting polar coordinates $\left(r,\theta\right)$ to rectangular coordinates $\left(x,y\right)$ the formula $x = r\cos\theta$ and $y = r\sin\theta$ are used where $-\pi < \theta \le \pi$. But the inverse conversion, rectangular $\left(x,y\right)$ to polar $\left(r,\theta\right)$ presents some problems since $\arctan\left(\frac{y}{x}\right)$ only returns $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and there is a possibility of division by zero if $x = 0$. To overcome these problems with the arctan function, the **atan2** function was developed and used in the programming language Fortran in 1961 for rectangular to polar conversions and by definition, $\theta = \text{atan2}\left(y,x\right)$ is the angle measured in radians with $-\pi < \theta \le \pi$ (Wikipedia 2023, Petzold 2008).

The atan2 function can also be expressed in terms of the standard arctan function as follows:

$$\text{atan2}\left(y,x\right) = \begin{cases} \arctan\left(\dfrac{y}{x}\right) & \text{if } x > 0, \\ \arctan\left(\dfrac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \ge 0, \\ \arctan\left(\dfrac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases} \tag{12}$$

Figure A4 below shows the four cases of $\theta = \text{atan2}\left(y,x\right)$ for $P\left(x,y\right)$ in quadrants I, II, III, and IV.

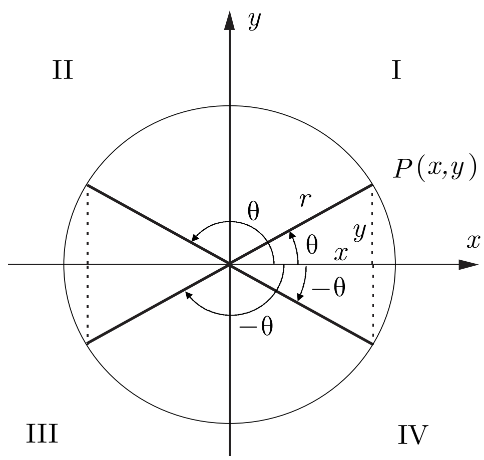

Figure A4. $\theta = \text{atan2}\left(y,x\right)$

Table A2 shows the values of $\theta = \text{atan2}\left(y,x\right)$ for $P\left(x,y\right)$ in quadrants I to IV dependent on positive or negative values of x and y.

Quadrant	$\left(y,x\right)$	$\theta = \text{atan2}\left(y,x\right)$
I	$\left(+,+\right)$	$+\theta$
II	$\left(+,-\right)$	$+\theta$
III	$\left(-,-\right)$	$-\theta$
IV	$\left(-,+\right)$	$-\theta$

Table A2. Values of $\text{atan2}\left(y,x\right)$ in quadrants

Determining bearings β using the arctan function

A bearing β is an angle measured 0 to 2π $\left(0^\circ \text{ to } 360^\circ\right)$ positive clockwise from the y axis. Bearings are used in surveying and mapping and navigation where the positive y axis coincides with North (magnetic, true, arbitrary) and the positive x axis coincides with East.

Now, in Figure A5, suppose $\alpha = \arctan\left(\frac{N}{E}\right)$ are angles $\left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right)$ measured positive anticlockwise from the E axis and bearings β $\left(0 \le \beta \le 2\pi\right)$ are positive clockwise from the N axis.

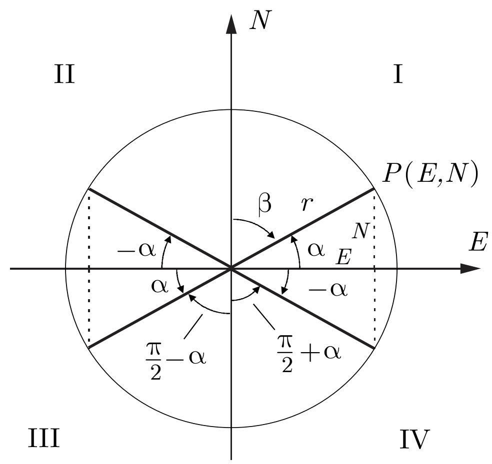

Figure A5.

For $P\left(E,N\right)$ where $E > 0$, $\beta = \pi - \left(\frac{\pi}{2} + \alpha\right) = \pi - \alpha - \frac{\pi}{2}$ and for $E < 0$, $\beta = \pi + \left(\frac{\pi}{2} - \alpha\right) = \pi - \alpha + \frac{\pi}{2}$ and combining these two relationships the bearing β is given by

$$\beta = \pi - \alpha - \operatorname{sgn}\left(E\right)\frac{\pi}{2} = \pi - \arctan\left(\frac{N}{E}\right) - \operatorname{sgn}\left(E\right)\frac{\pi}{2} \qquad \text{for } E \ne 0 \tag{13}$$

Where sgn is the *signum* or *sign* function and $\operatorname{sgn}\left(x\right) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$

For calculations, to guard against a possible division by zero using the arctan function, a very small quantity say $\varepsilon = 1E-10$ can be added to E and (13) can be written as

$$\beta = \pi - \alpha - \operatorname{sgn}\left(E + \varepsilon\right)\frac{\pi}{2} = \pi - \arctan\left(\frac{N}{E + \varepsilon}\right) - \operatorname{sgn}\left(E + \varepsilon\right)\frac{\pi}{2} \tag{14}$$

Determining bearings β using the atan2 function

Similarly to the previous section, in Figure A6, suppose $\alpha = \operatorname{atan2}\left(-E, -N\right)$ are angles $\left(-\pi < \alpha \le \pi\right)$ measured positive clockwise from the S axis (S or south is the opposite direction to N or north), noting here that swapping the signs of E and N in the atan2 function changes the definition of the angles (see Figure A4). Bearings β $\left(0 \le \beta \le 2\pi\right)$ are positive clockwise from the N axis.

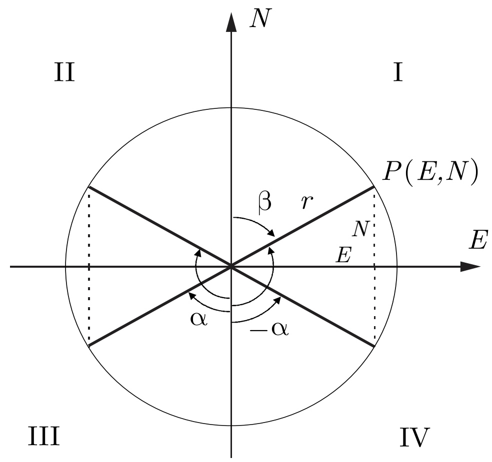

Figure A6.

For $P\left(E, N\right)$ where $E > 0$, $\beta = \pi + \alpha$ and for $E < 0$, $\beta = \pi + \alpha$ hence

$$\beta = \pi + \alpha = \pi + \operatorname{atan2}\left(-E, -N\right) \tag{15}$$

Notation for atan2 in Excel

Many scientific software packages and computer languages define an atan2 function. Most use the form of notation shown here, i.e., in the first quadrant $\arctan\left(\frac{y}{x}\right) = \operatorname{atan2}\left(y, x\right)$ for $x \ne 0$. But some do not, and in particular Microsoft Excel, that has the following definition: $\arctan\left(\frac{b}{a}\right) = \operatorname{atan2}\left(a, b\right)$ for $a \ne 0$